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Abstract

We consider a Ginzburg–Landau functional for a complex vector order parameter $\Psi = (\psi_+, \psi_-)$, whose minimizers exhibit vortices with half-integer degree. By studying the associated system of equations in $\mathbf{R}^2$ which describes the local structure of these vortices, we show some new and unconventional properties of these vortices. In particular, one component of the solution vanishes, but the other does not. We also prove the existence and uniqueness of equivariant entire solutions, and provide a second proof of uniqueness, valid for a large class of systems with variational structure.

1 Introduction

Recent papers in the physics literature have introduced spin-coupled (or spinor) Ginzburg–Landau models for complex vector-valued order parameters in order to account for ferromagnetic (or antiferromagnetic) effects in high-temperature superconductors [KR] and in optically confined Bose–Einstein condensates [IM]. In [AB2] two of the authors studied one such model, for a complex pair of order parameters and showed that minimizers exhibit new types of vortices, with fractional degrees. In this paper we consider the structure of these fractional degree vortices, and show that their cores are qualitatively different from Ginzburg–Landau vortices.

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Consider the following model problem, related to the superconductivity model introduced in [KR]. Let $\Omega \subset \mathbf{R}^2$ be a smooth, bounded domain, and $\Psi \in H^1(\Omega;\mathbf{C}^2)$. We define an energy functional,

$$E_\epsilon(\Psi) = \frac{1}{2}\int_\Omega \left\{ |\nabla\Psi|^2 + \frac{1}{2\epsilon^2}\left(|\Psi|^2 - 1\right)^2 + \frac{2\gamma}{\epsilon^2}\left(\psi_1\times\psi_2\right)^2 \right\} dx,$$

where $\Psi = (\psi_1,\psi_2)$, $\psi_1\times\psi_2 = \mathrm{Im}\,(\overline{\psi_1}\psi_2)$, $\gamma > 0$ and $\epsilon > 0$ are parameters. The quantity

$$S = \psi_1\times\psi_2 = \mathrm{Im}\,\{\overline{\psi}_1\,\psi_2\}$$

is interpreted as the z-component of a spin vector, which in this two-dimensional model is assumed to be orthogonal to the plane of Ω.

As $\epsilon \to 0$, energy minimizers should converge pointwise to the manifold on which the potential term $F(\Psi) = \left(|\Psi|^2 - 1\right)^2 + \frac{\gamma}{2}\left(\psi_1\times\psi_2\right)^2$ vanishes. Since $\gamma > 0$, we obtain a two-dimensional surface (a 2-torus) $\Sigma \subset S^3 \subset \mathbf{C}^2$ parametrized by two real phases, ϕ,ω:

$$\Sigma: \quad \Psi = G(\phi,\omega) := (e^{i\phi}\cos\omega\,,\; e^{i\phi}\sin\omega).$$

Notice that G is doubly-periodic with minimal period $G(\phi+\pi,\omega\pm\pi) = G(\phi,\omega)$, with each phase executing a *half* cycle. For a smooth function $\Psi(x)$ taking values in Σ and a simple closed curve C contained in the domain of Ψ we may therefore define a pair of *half-integer valued* degrees (d_ϕ, d_ω) corresponding to the winding numbers of the two phases around Σ. From the above observation, these degrees satisfy $d_\phi, d_\omega \in \frac{1}{2}\mathbf{Z}$, and $d_\phi + d_\omega \in \mathbf{Z}$.

When auxilliary conditions force one or the other of the two phases ϕ,ω to have nontrivial winding number the minimizer $\Psi(x)$ cannot take values in Σ at every point in Ω and in the limit we observe *vortices*, just as in the classical Ginzburg–Landau model. Each isolated vortex will carry a pair of half-integer degrees, (d_ϕ, d_ω) as above.

The results of [AB2] describe the minimizers and their energies as $\epsilon \to 0$, with a given Dirichlet boundary condition $\Psi|_{\partial\Omega} = g$, where $g = (g_1, g_2)$ is a given smooth function $g: \partial\Omega \to \Sigma$. The boundary condition admits degrees D_ϕ, D_ω corresponding to its winding in each of the phases around $\partial\Omega$. Assume for simplicity that $D_\phi \ge |D_\omega|$. The main theorem of [AB2] then states that the minimizers can exhibit vortices of three different topological types: two species of fractional degree vortices, $(d_\phi, d_\omega) = (\frac{1}{2},\frac{1}{2})$ or $(\frac{1}{2},-\frac{1}{2})$, and an integer-degree vortex, $(d_\phi, d_\omega) = (1,0)$. The integer-degree vortex can be seen as a superposition of the two different fractional-degree vortices at the same location in Ω, and indeed the energy expansion shows that there is a weak interaction which favors the combination of two nearby fractional-degree vortices into a single $(1,0)$-vortex.

We expect, however, that these distinct types of vortices are very different in their microscopic structure. In order to resolve the singularity at each vortex, the order parameter Ψ must deviate from the minimal manifold $\Sigma \subset \mathbf{C}^2$. The surface Σ being of codimension two, there are two degrees of freedom for this to occur. The order parameter can choose to violate the condition $|\Psi| = 1$ and develop a zero at the core of the vortex, as is the case for the usual Ginzburg–Landau vortices. But there is another possibility: Ψ can rotate along the sphere $|\Psi| = 1$ and violate the condition $S = 0$, thus acquiring non-zero spin in its core and avoiding the vanishing of $|\Psi|$ altogether. The integer-degree $(d_\phi, d_\omega) = (1, 0)$ vortices will take the first option, and resemble usual Ginzburg–Landau vortices, but our results in this paper confirm that the two species of fractional-degree vortices will indeed prefer the second approach, and exhibit this new "coreless" structure. To do this, we blow up the minimizers around each vortex and study the associated limiting problem of entire solutions to the PDE system in the whole of $\mathbf{R}^2$, using techniques introduced for the Ginzburg–Landau equation by Brezis, Merle, & Rivière [BMR], Mironescu [M], and Shafrir [S].

In order to describe our results, we introduce a change of variable as in [AB1], [AB2] which simplifies the accounting of degrees. Vortices are best described in terms of the (integer) indices $[n_+, n_-]$,

$$n_+ = d_\phi + d_\omega, \quad n_- = d_\phi - d_\omega, \quad (d_\phi, d_\omega) = n_+(1/2\,,\,1/2) + n_-(1/2\,,\,-1/2),$$

which count the number of these two species of fractional-degree vortices rather than their winding. Remarkably, this may be achieved via the linear transformation in the range,

$$\psi_\pm := \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_2).$$

In the new coordinates we denote our order parameter as $\Psi = [\psi_+, \psi_-]$. Now the surface Σ is described more simply,

$$\Sigma : \quad |\psi_+|^2 = \frac{1}{2} = |\psi_-|^2,$$

and is parametrized as

$$\Psi = \left[\frac{1}{\sqrt{2}}e^{i\alpha_+}, \frac{1}{\sqrt{2}}e^{i\alpha_-}\right]$$

with phases $\alpha_\pm$ carrying whole number degrees $[n_+, n_-]$. Note the following correspondences between the degrees (d_ϕ, d_ω) and the integer indices $[n_+, n_-]$:

$$(d_\phi, d_\omega) = \left(\frac{1}{2}, \frac{1}{2}\right) \leftrightarrow [n_+, n_-] = [1, 0], \qquad (d_\phi, d_\omega) = \left(\frac{1}{2}, -\frac{1}{2}\right) \leftrightarrow [n_+, n_-] = [0, 1],$$
$$(d_\phi, d_\omega) = (1, 0) \quad \leftrightarrow \quad [n_+, n_-] = [1, 1].$$

In these coordinates, the spin is given by $S = \frac{1}{2}(|\psi_-|^2 - |\psi_+|^2)$.

The equations for entire vortex solutions $\Psi(x) = [\psi_+, \psi_-]$ then become,

$$-\Delta\psi_+ = (1 - |\Psi|^2)\psi_+ + \gamma(|\psi_-|^2 - |\psi_+|^2)\psi_+, \tag{1}$$ $\boxed{\texttt{Eq+}}$

$$-\Delta\psi_- = (1 - |\Psi|^2)\psi_- - \gamma(|\psi_-|^2 - |\psi_+|^2)\psi_-. \tag{2}$$ $\boxed{\texttt{Eq-}}$

Solutions to (1–2) obtained by blowing up will satisfy an integrability condition,

$$\int_{\mathbf{R}^2} \left\{ \left(|\Psi|^2 - 1\right)^2 + \gamma \left(|\psi_-|^2 - |\psi_+|^2\right)^2 \right\} dx < \infty, \tag{3}$$ $\boxed{\texttt{BMRcond}}$

analogous to the condition of [BMR] for the classical Ginzburg–Landau equation. It follows from arguments of [BMR] (see Lemma 2.3) that any solution satisfying (3) has a degree pair at infinity: $n_\pm = \deg\left(\frac{\psi_\pm}{|\psi_\pm|}; S_R\right)$ (with S_R the circle of radius R,) for all sufficiently large radii R. As was the case for the classical Ginzburg–Landau equations [BMR], the integral in (3) is quantized:

$$\int_{\mathbf{R}^2} \left\{ \left(|\Psi|^2 - 1\right)^2 + \gamma \left(|\psi_-|^2 - |\psi_+|^2\right)^2 \right\} dx = \pi \left(n_+^2 + n_-^2\right). \tag{4}$$ $\boxed{\texttt{quantized}}$

This fact, together with some asymptotic estimates of the behavior of solutions at infinity, will be proven in Proposition 2.1.

We would like to relate solutions of (1–2) to energy minimization. If $\Omega \subset \mathbf{R}^2$ is a bounded domain, we may define an energy locally by

$$E(\Psi; \Omega) = \int_\Omega \left\{ \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{4}\left(|\Psi|^2 - 1\right)^2 + \frac{\gamma}{4}\left(|\psi_-|^2 - |\psi_+|^2\right)^2 \right\} dx \tag{5}$$ $\boxed{\texttt{Energy}}$

This energy diverges when either of ψ_+, ψ_- has nontrivial winding number at infinity, so it is not well-defined when $\Omega = \mathbf{R}^2$. Instead, we define locally minimizing solutions in $\mathbf{R}^2$ in the sense of De Giorgi: we say that Ψ is a locally minimizing solution of (1–2) if (3) holds and if for every bounded regular domain $\Omega \subset \mathbf{R}^2$,

$$E(\Psi; \Omega) \le E(\Phi; \Omega)$$

holds for every $\Phi = [\varphi_+, \varphi_-] \in H^1(\Omega; \mathbf{C}^2)$ with $\Phi|_{\partial\Omega} = \Psi|_{\partial\Omega}$. We prove the following result concerning fractional-degree vortex solutions:

$\boxed{\texttt{locminthm}}$ **Theorem 1.1** *Suppose Ψ is a locally minimizing solution of (1–2) with degree pair $[n_+, n_-] = [1, 0]$. Then there exists a constant $\phi_- \in [0, 2\pi)$ such that $\psi_-(x)e^{i\phi_-} > 0$ is real and positive in $\mathbf{R}^2$.*

In particular, the ψ_--component of the order parameter is bounded away from zero in $\mathbf{R}^2$, and the qualitative behavior is as we expected, with $|\Psi|$ bounded away from zero and $S = \frac{1}{2}(|\psi_-|^2 - |\psi_+|^2) > 0$ in the core. An analogous result is obtained for the $[n_+, n_-] = [0,1]$-vortex, except that now it is ψ_+ which is bounded away from zero and the spin $S < 0$ in the core. By a straightforward argument, solutions obtained by blowing up minimizers of E_ϵ around a vortex always yield local minimizers in the sense of De Giorgi, and hence we infer the following result:

epsthm **Theorem 1.2** *Assume $\Psi_\epsilon = [\psi_{\epsilon+}, \psi_{\epsilon-}]$ is a family of minimizers of E_ϵ as $\epsilon \to 0$. If $p \in \Omega$ is such that Ψ_ϵ converges in some deleted neighborhood $B_\delta(p) \setminus \{p\}$ to a canonical Σ-harmonic map with degrees $[n_+, n_-] = [1,0]$ at p, then for small ϵ, $|\psi_{\epsilon-}(x)|$ is bounded away from zero in $B_\delta(p)$.*

Special solutions to (1–2) are obtained by an equivariant ansatz, $\psi_\pm(x) = f_\pm(r)e^{in_\pm\theta}$, in polar coordinates (r,θ) in $\mathbf{R}^2$, with $f_\pm$ a pair of real-valued functions. In section 3 we show that for each fixed choice of degrees $n_\pm$ at infinity, there exist unique equivariant entire solutions satisfying (3). Uniqueness is proven using the method of Brezis & Oswald [BO]. In section 4 we give an alternative proof of uniqueness of equivariant solutions by means of an extension of the Krasnoselskii theorem [K] to systems with variational structure.

We also study the interesting role of the parameter γ in the monotonicity of the fractional-degree vortex profiles. When $0 < \gamma < 1$ the component of the order parameter which does *not* vanish (for example, f_- for the $[n_+, n_-] = [1,0]$ vortex) is monotone *decreasing*, and approaches its limiting value at infinity *from above.* When $\gamma \geq 1$, all vortices of any degree combination have density profiles which *increase* with r, just as in the Ginzburg–Landau case.

monotone **Theorem 1.3** *Assume $\Psi(x) = [f_+(r)e^{in_+\theta}, f_-(r)e^{in_-\theta}]$ is an equivariant solution satisfying (3).*

(i) If $\gamma \geq 1$, then $f_\pm'(r) \geq 0$ for all $r > 0$, for any degrees $[n_+, n_-]$.

(ii) If $0 < \gamma < 1$, $n_+ \geq 1$, and $n_- = 0$, then $f_+'(r) \geq 0$ and $f_-'(r) \leq 0$ for all $r > 0$.

The methods used in Section 3 are derived from the work of Alama, Bronsard & Giorgi [ABG1, ABG2] on the $SO(5)$-model, which also featured a vector-valued order parameter and two different species of vortex profiles.

A natural open question is whether all locally minimizing solutions to (1–2) must be radial. This fact was proven by Mironescu [M] for the classical Ginzburg–Landau equation, by dividing any given solution by an equivariant one (which must be of degree ± 1,) and calculating a sort of Pohozaev identity for the equation satisfied by the quotient. Our equations being a coupled system, the above procedure fails since the equation for the quotient is no longer clearly of gradient form. Although our system is in many ways very similar to the scalar Ginzburg–Landau equation, and many analytic results may be extended from one to the other, we have been careful in verifying which techniques derived for the scalar equation may be adapted to the system (1–2).

2 Locally minimizing solutions

In this section we study solutions which are locally minimizing in the sense of De Giorgi, and prove Theorem 1.1. The proof is based on the following asymptotic description of solutions:

asympt **Proposition 2.1** *Let Ψ be a solution of (1–2) in $\mathbf{R}^2$ satisfying (3). There exist constants β_+, β_- such that $\psi_\pm \to \frac{1}{\sqrt{2}} e^{i(n_\pm \theta + \beta_\pm)}$ uniformly as $|x| \to \infty$. Moreover:*

(i) If $[n_+, n_-] = [1, 0]$, then as $r = |x| \to \infty$,

$$|\psi_+(x)|^2 = \frac{1}{2} - \frac{\gamma+1}{4\gamma}\frac{1}{r^2} + o\left(\frac{1}{r^2}\right), \quad |\psi_-(x)|^2 = \frac{1}{2} - \frac{\gamma-1}{4\gamma}\frac{1}{r^2} + o\left(\frac{1}{r^2}\right). \tag{6}$$ asympt10

(ii) If $[n_+, n_-] = [1, 1]$, then as $r = |x| \to \infty$,

$$|\psi_\pm(x)|^2 = \frac{1}{2} - \frac{1}{2r^2} + o\left(\frac{1}{r^2}\right). \tag{7}$$ asympt11

asymptrem **Remark 2.2** We observe that this asymptotic result already shows the qualitative difference between the case $0 < \gamma < 1$ and the case $\gamma \geq 1$, at least in the case of the fractional degree vortices. We will confirm this difference in monotonicity of the vortex profiles in our analysis of the equivariant (radially symmetric) vortex solutions in the next section.

Proof of Theorem 1.1: By the first part of Proposition 2.1, we may assume without loss of generality that $\psi_-(x) \to \frac{1}{\sqrt{2}}$ uniformly as $|x| \to \infty$. In particular, if we fix any $\delta < \frac{1}{2\sqrt{2}}$, there exists a radius $R = R(\delta)$ such that $|\psi_-(x) - \frac{1}{\sqrt{2}}| < \delta$ for all $|x| \geq R$. Let $\Omega = B_R(0)$, and for $x \in \Omega$ define

$$\tilde{\psi}_+(x) = \psi_+(x), \quad \tilde{\psi}_-(x) = |\mathrm{Re}\, \psi_-(x)| + i\, \mathrm{Im}\, \psi_-(x).$$

Note that $\tilde{\Psi} := [\tilde{\psi}_+, \tilde{\psi}_-] \in H^1(\Omega; \mathbf{C}^2)$, $E(\tilde{\Psi}; \Omega) = E(\Psi; \Omega)$, and (by the choice of R,) $\tilde{\Psi}|_{\partial\Omega} = \Psi|_{\partial\Omega}$. Therefore, $\tilde{\Psi}$ is also a local minimizer of E, in the sense described above. This implies that $\tilde{\Psi}$ also solves the Euler–Lagrange equations (1–2) in Ω. In particular, $u = \mathrm{Re}\,\tilde{\psi}_-$ is a non-negative solution of

$$-\Delta u + \Big((1-\gamma)|\tilde{\psi}_+|^2 + (1+\gamma)|\tilde{\psi}_-|^2 - 1\Big)\, u = 0,$$

which is strictly positive on $\partial\Omega = S_R$ (again, by the choice of R.) By the Strong Maximum Principle, in fact $u = \mathrm{Re}\,\tilde{\psi}_-(x) > 0$ in Ω. This implies $\tilde{\Psi} = \Psi$, and

$$\mathrm{Re}\,\psi_-(x) > 0 \quad \text{in } \mathbf{R}^2.$$

Now let α be a constant with $|\alpha| < \frac{\pi}{2}$, and consider $\hat{\psi}_-(x) := \psi_-(x)e^{i\alpha}$. Note that $\hat{\Psi} := [\psi_+, \hat{\psi}_-]$ is again a solution to (1–2), with the same energy in any domain Ω. By Proposition 2.1 and our definition of $\hat{\psi}_-$ we now have $\hat{\psi}_-(x) \to \frac{1}{\sqrt{2}}e^{i\alpha}$ uniformly as $|x| \to \infty$. Choosing $\delta = \delta(\alpha) > 0$ such that $B_\delta(\frac{1}{\sqrt{2}}e^{i\alpha})$ is strictly contained inside the right half-plane $\{\mathrm{Re}\, z > 0\}$, there exists a radius $R = R(\alpha)$ such that $|\hat{\psi}_-(x) - \frac{1}{\sqrt{2}}e^{i\alpha}| < \delta$ whenever $|x| \geq R$. Repeating the above argument, we conclude that $\mathrm{Re}\,\hat{\psi}_-(x) > 0$ in $\mathbf{R}^2$. Equivalently,

$$\begin{aligned} \mathrm{Im}\,\psi_-(x) &\leq (\cot\alpha)\,\mathrm{Re}\,\psi_-(x) \qquad \text{when } 0 < \alpha < \pi/2, \\ \mathrm{Im}\,\psi_-(x) &\geq (\cot\alpha)\,\mathrm{Re}\,\psi_-(x) \qquad \text{when } -\pi/2 < \alpha < 0. \end{aligned}$$

Letting $\alpha \to \pm\frac{\pi}{2}$ we conclude $\mathrm{Im}\,\psi_-(x) \equiv 0$.
$\diamondsuit$

To prove Proposition 2.1 we use the following modification of a similar result from [BMR]:

bmr **Lemma 2.3** *Let Ψ be an entire solution of (1–2) satisfying (3).*

(i) $|\Psi(x)| \leq 1$ for all $x \in \mathbf{R}^2$ and $|\psi_\pm(x)|^2 \to \frac{1}{2}$ uniformly as $|x| \to \infty$.

(ii) There exist constants $R_0 > 0$, $n_\pm \in \mathbf{Z}$, and smooth functions $\rho_\pm(x), \phi_\pm(x)$ for $|x| \geq R_0$ such that

$$\Psi(x) = [\psi_+(x), \psi_-(x)] = \left[\rho_+(x)e^{i(n_+\theta+\phi_+(x))}, \rho_-(x)e^{i(n_-\theta+\phi_-(x))}\right],$$

with

$$\int_{|x|>R_0} \Big(|\nabla\rho_\pm|^2 + |\nabla\phi_\pm|^2\Big) < \infty. \qquad (8)$$ bmrest

Proof: Statement (i) follows as in Step 1 of the proof of Theorem 1 in [BMR]. Indeed, the quantity $U(x) := |\Psi(x)|^2$ satisfies the equation

$$\frac{1}{2}\Delta U = (U-1)u + \frac{\gamma}{2}S^2 + |\nabla\Psi|^2,$$

and hence the estimate $U = |\Psi|^2 < 1$ follows from the strong maximum principle. The uniform convergence as $|x| \to \infty$ then follows as in [BMR] from standard elliptic estimates, (3), and the following elementary inequality [AB2]:

$$2\min\{1,\gamma\}\left[\left(|\psi_+|^2 - \frac{1}{2}\right)^2 + \left(|\psi_-|^2 - \frac{1}{2}\right)^2\right] \leq \left(|\Psi|^2 - 1\right)^2 + 4\gamma S^2.$$

The existence of R_0, $\rho_\pm$, $\psi_\pm$ is an immediate consequence of (i). The first part of (ii) is an immediate consequence of the uniform limit $|x| \to \infty$. To prove (8), we write the equations for $\varphi_\pm = n_\pm\theta + \phi_\pm$ and $\rho_\pm$:

$$\begin{aligned} \operatorname{div}\left(\rho_\pm^2 \nabla\varphi_\pm\right) &= 0, \\ -\Delta\rho_\pm + |\nabla\varphi_\pm|^2\rho_\pm &= (1 - \rho_+^2 - \rho_-^2)\rho_\pm \mp \gamma(\rho_-^2 - \rho_+^2)\rho_\pm. \end{aligned}$$

The equations for $\varphi_\pm$ are identical to those in [BMR], and the analysis there applies with no modification. The equations for $\rho_\pm$ are of the same form, and the same approach as [BMR] leads easily to the same conclusion with only minor changes. We leave the details to the interested reader.

$\diamondsuit$

Proof of Proposition 2.1: The proof follows Shafrir [S]. Let $R_m \to \infty$ be any increasing divergent sequence, $\epsilon_m = 1/R_m$, and let $0 < a < 1 < b$ be fixed. Denote by $\Omega = B_b(0)\setminus\overline{B_a(0)}$ and $\Omega_m = B_{bR_m} \setminus \overline{B_{aR_m}(0)}$. Consider the rescaled solutions

$$\Psi_m(x) = [\psi_{m+}(x), \psi_{m-}(x)] = \Psi(R_m x).$$

Then Ψ_m satisfies

$$-\Delta\psi_{m\pm} + \frac{1}{\epsilon_m^2}\left((1\pm\gamma)|\psi_{m+}|^2 + (1\mp\gamma)|\psi_{m-}|^2 - 1\right)\psi_{m\pm} = 0 \quad \text{in } \Omega. \tag{9}$$ singpert

We now apply Lemma 2.3 to obtain $R_0 > 0$ and $\rho_\pm$, $\phi_\pm$ defined for $|x| \geq R_0$. Since large $|x|$ is equivalent to large m we may write, for large m, $\psi_{m\pm} = \rho_{m\pm}\exp(i(n_\pm\theta + \phi_{m\pm}(x)))$. As in [S], we use (8) to calculate

$$\begin{aligned} \int_\Omega |\nabla\Psi_m|^2 &= \int_{\Omega_m} |\nabla\Psi|^2 = \int_{\Omega_m}\sum_\pm\left[|\nabla\rho_\pm|^2 + \rho_\pm^2|n_\pm\nabla\theta + \nabla\phi_\pm|^2\right] \\ &= \int_{\Omega_m}\sum_\pm\left[|\nabla\rho_\pm|^2 + \rho_\pm^2\left(\frac{n_\pm^2}{r^2} + \frac{2n_\pm}{r^2}\nabla\phi_\pm\cdot(-y,x) + |\nabla\phi_\pm|^2\right)\right] \\ &= \int_{\Omega_m}\sum_\pm\frac{1}{2}\frac{n_\pm^2}{r^2} + o(1) \\ &= \pi\left(n_+^2 + n_-^2\right)\ln\frac{b}{a} + o(1). \end{aligned} \tag{10}$$ upper

Up to a subsequence, we find $\Psi_m \rightharpoonup \tilde\Psi$ in $H^1(\Omega;\Sigma)$.

We claim that the convergence $\Psi_m \to \tilde\Psi$ is strong in $H^1(\Omega;\Sigma)$, and

$$\tilde\Psi(x) = \frac{1}{\sqrt{2}}[e^{i(n_+\theta+\beta_+)}, e^{i(n_+\theta+\beta_-)}], \tag{11}$$ Psitilde

with $\beta_\pm$ real constants. Indeed, since $\tilde\Psi$ takes values in Σ we may represent it locally as $\tilde\Psi = \frac{1}{\sqrt{2}}[\exp(i\varphi_+(x)),\, \exp(i\varphi_-(x))]$, where $\varphi_\pm$ are possibly multivalued, real-valued functions. By standard arguments we derive a lower bound which matches (10):

$$\begin{aligned}\int_\Omega |\nabla\tilde\psi_\pm|^2 &\geq \int_a^b \int_0^{2\pi} \frac{1}{2}\left(\nabla\varphi_\pm\cdot\hat\theta\right)^2 r\,d\theta\,dr \\ &\geq \frac{1}{2}\int_a^b \frac{\left[\int_{S_r}\frac{\partial\varphi_\pm}{\partial s}\,ds_r\right]^2}{\int_0^{2\pi} 1\, r\,d\theta}\,dr \\ &= \int_a^b \frac{\pi n_\pm^2}{r}\,dr = \pi\, n_\pm^2\,\ln\frac{b}{a}.\end{aligned}$$

By lower semicontinuity, we conclude that this inequality is indeed an equality, $\int_\Omega |\nabla\tilde\Psi|^2 = \pi(n_+^2+n_-^2)\ln\frac{b}{a}$. Hence, the convergence is strong in H^1. In addition, we have the case of equality in the Cauchy-Schwarz inequality used in the second line of the lower bound above, which implies (11), and the claim is established.

We now employ the main idea of [S]: to use the local convergence results away from vortices for the singularly perturbed problem (9), derived for the Ginzburg–Landau equation in [BBH1] and extended to our spinor system in [AB2]. By Theorem 4.1 in [AB2], $\Psi_m \to \tilde\Psi$ in $C^k_{loc}(\Omega)$ for any $k\geq 0$, and

$$\left\|\frac{1}{\epsilon_m^2}\left\{(1\pm\gamma)|\tilde\psi_{m+}|^2 + (1\mp\gamma)|\tilde\psi_{m-}|^2 - 1\right\} + 2|\nabla\tilde\psi_{m\pm}|^2\right\|_{C^k_{loc}(\Omega)} \to 0, \qquad \text{for all } k\geq 0.$$

Note that the C^k_{loc} convergence of Ψ_m to $\tilde\Psi$ implies that we may replace $2|\nabla\tilde\psi_{m\pm}|^2$ by $\frac{n_\pm^2}{r^2}$ in the above estimate. Evaluating along $\partial B_1(0)\subset\Omega$,

$$\left\|R_m^2\left\{(1\pm\gamma)|\psi_+(R_m x)|^2 + (1\mp\gamma)|\psi_-(R_m x)|^2 - 1\right\} + n_\pm^2\right\|_{L^\infty(\partial B_1(0))} \to 0.$$

Since R_m was an arbitrary divergent sequence we may conclude that the above holds for general $r\to\infty$, that is,

$$\left\{(1\pm\gamma)|\psi_+(x)|^2 + (1\mp\gamma)|\psi_-(x)|^2 - 1\right\} + \frac{n_\pm^2}{r^2} = o\left(\frac{1}{r^2}\right),$$

uniformly as $|x| = r\to\infty$. This then implies that

$$|\psi_+|^2 = \frac{1}{2} - \frac{n_+^2(\gamma+1)+n_-^2(\gamma-1)}{4\gamma}\,\frac{1}{r^2} + o\left(\frac{1}{r^2}\right),$$
$$|\psi_-|^2 = \frac{1}{2} - \frac{n_+^2(\gamma-1)+n_-^2(\gamma+1)}{4\gamma}\,\frac{1}{r^2} + o\left(\frac{1}{r^2}\right),$$

as $r \to \infty$. The conclusions (6) and (7) then follow immediately.

To obtain the uniform limit of $\phi_\pm(x)$, we note that by taking the imaginary part of the equations (1–2) in polar form we arrive at the same equation (for conservation of current) as in the classical Ginzburg–Landau equation,

$$\mathrm{div}\, \left(\rho_\pm^2 \nabla (n_\pm \theta + \phi_\pm)\right) = 0.$$

Therefore the assertion that $\phi_\pm(x) \to \beta_\pm$ uniformly as $|x| \to \infty$ follows exactly as in [S].
$\diamondsuit$

We note the following further estimate which will be useful in our study of equivariant solutions in the next section:

derivs **Corollary 2.4** *Under the hypotheses above, with* $\rho_\pm = |\psi_\pm|$, *we have:*

$$\begin{aligned} \frac{\partial \rho_\pm}{\partial r} &= \frac{n_+^2(\gamma \pm 1) + n_-^2(\gamma \mp 1)}{2\sqrt{2}\gamma} \frac{1}{r^3} + o\left(\frac{1}{r^3}\right) \\ \frac{\partial^2 \rho_\pm}{\partial r^2} &= -\frac{3\sqrt{2}}{4\gamma}\left[n_+^2(\gamma \pm 1) + n_-^2(\gamma \mp 1)\right] \frac{1}{r^4} + o\left(\frac{1}{r^4}\right). \end{aligned}$$

These follow by differentiation in the C^k_{loc} estimates above.

Finally, we prove the quantization of the potential term for any entire solution satisfying (3):

BMRprop **Proposition 2.5** *Let (for any choice of* $[n_+, n_-]$*)* $\Psi = [\psi_+, \psi_-]$ *be a solution of (1–2) satisfying (3). Then*

$$\int_{\mathbf{R}^2} \left\{ \left(|\Psi|^2 - 1\right)^2 + \gamma \left(|\psi_-|^2 - |\psi_+|^2\right)^2 \right\} dx = \pi \left(n_+^2 + n_-^2\right).$$

Proof: The proof continues from that of Lemma 2.3, following Step 3 in the proof of Theorem 1 in [BMR]. By the Pohozaev identity applied to our system,

$$\frac{1}{r}\int_{B_r} G(\Psi)\, dx + \int_{\partial B_r} \left[\left|\frac{\partial \psi_+}{\partial r}\right|^2 + \left|\frac{\partial \psi_-}{\partial r}\right|^2 \right] ds = \int_{\partial B_r} \left[\left|\frac{\partial \psi_+}{\partial \tau}\right|^2 + \left|\frac{\partial \psi_-}{\partial \tau}\right|^2 + G(\Psi) \right] ds, \quad (12) \quad \text{Po}$$

where τ indicates the unit tangent to ∂B_r and

$$G(\Psi) := \left(|\Psi|^2 - 1\right)^2 + \gamma \left(|\psi_-|^2 - |\psi_+|^2\right)^2.$$

Define

$$E(R) := \int_{B_R} G(\Psi)\, dx, \qquad E := \int_{\mathbf{R}^2} G(\Psi)\, dx,$$

and note that $E(R)\to E$ as $R\to\infty$, as well as

$$\frac{1}{\ln R}\int_0^R \frac{E(r)}{r}dr \to E, \qquad R\to\infty.$$

Integrating (12) over $r\in(0,R)$,

$$\int_{B_R}\left[\left|\frac{\partial\psi_+}{\partial r}\right|^2+\left|\frac{\partial\psi_-}{\partial r}\right|^2\right]+\int_0^R\frac{E(r)}{r}dr=\int_{B_R}\left[\left|\frac{\partial\psi_+}{\partial\tau}\right|^2+\left|\frac{\partial\psi_-}{\partial\tau}\right|^2\right]+\frac12 E(R). \tag{13}$$ Po1

The radial derivatives $|\frac{\partial\psi_\pm}{\partial r}|^2$ are estimated as in (2.46) of [BMR], using (8) to obtain

$$\int_{B_R}\left[\left|\frac{\partial\psi_+}{\partial r}\right|^2+\left|\frac{\partial\psi_-}{\partial r}\right|^2\right]\le C$$

uniformly as $R\to\infty$. The difference in our case is in the tangential derivative,

$$\left|\left|\frac{\partial\psi_\pm}{\partial\tau}\right|^2-\frac{n_\pm^2}{2r^2}\right| \;\le\; |\nabla\rho_\pm|^2+\left|\rho^2-\frac12\right|\frac{n_\pm^2}{r^2}+2\rho_\pm^2\frac{n_\pm}{r}|\nabla\phi_\pm|+|\nabla\phi_\pm|^2,$$

where we have decomposed $\psi_\pm(x)=\rho_\pm\, e^{i(n_\pm\theta+\phi_\pm)}$ as in the proof of Lemma 2.3. Note the extra factor $\frac12$ which appears in our case since $\rho_\pm=|\psi_\pm|\to 1/\sqrt2$ as $|x|\to\infty$. Continuing as in (2.49), (2.50) of [BMR], we divide (13) by $\ln R$ and pass to the limit to obtain:

$$\int_{\mathbf{R}^2}G(\Psi)\,dx=\lim_{R\to\infty}\frac{1}{\ln R}\int_{B_R}\left(\left|\frac{\partial\psi_+}{\partial\tau}\right|^2+\left|\frac{\partial\psi_-}{\partial\tau}\right|^2\right)=\lim_{R\to\infty}\frac{1}{\ln R}\int_{B_R}\frac{n_\pm^2}{2r^2}dx=\pi(n_+^2+n_-^2).$$

$\diamondsuit$

3 Equivariant solutions

In this section we consider special solutions of the equations (1–2) of the form

$$\psi_+(x)=f_+(r)\,e^{in_+\theta},\quad \psi_-(x)=f_-(r)\,e^{in_-\theta},$$

in polar coordinates (r,θ) with given degree pair $[n_+,n_-]\in\mathbf{Z}^2$. By taking complex conjugates if necessary, we may assume that $n_\pm\ge 0$. When $f\ge 0$, the system (1–2) reduces to the following system of ODEs,

$$\left.\begin{array}{c} -\frac1r\left(rf_\pm'\right)'+\frac{n_\pm^2}{r^2}f_\pm+\left(f_+^2+f_-^2-1\right)f_\pm\ \mp\gamma\left(f_-^2-f_+^2\right)f_\pm=0,\ \text{for } r\in(0,\infty),\\[2ex] f_\pm(r)\ge 0 \qquad \text{for all } r\in[0,\infty),\\[2ex] f_\pm(R)\to\frac{1}{\sqrt2}\ \text{as } r\to\infty,\\[2ex] f_\pm(0)=0 \text{ if } n_\pm\ne 0,\ f_\pm'(0)=0 \text{ if } n_\pm=0. \end{array}\right\} \tag{14}$$ ODE

We begin with their existence and uniqueness:

exist **Lemma 3.1** *Let $n_\pm \in \mathbf{Z}$ be given. Then there exists a unique solution $[f_+(r), f_-(r)]$ to (14) for $r \in [0,\infty)$ such that: $f_\pm \in C^\infty((0,\infty))$, $f_\pm(r) > 0$ for all $r > 0$, $\int_0^\infty (1 - f_+^2 - f_-^2)^2\, r\, dr < \infty$, and $f_\pm(r) \sim r^{n_\pm}$ for $r \sim 0$. In particular, $\Psi(x) = [f_+(r)e^{in_+\theta}, f_-(r)e^{in_-\theta}]$ is an entire solution of (1–2) in $\mathbf{R}^2$ satisfying (3).*

Proof: To obtain existence we consider first the simpler problem defined in the ball B_R, $R > 0$,

$$\left.\begin{aligned} -\tfrac{1}{r}\left(r f_\pm'\right)' + \tfrac{n_\pm^2}{r^2} f_\pm + \left(f_+^2 + f_-^2 - 1\right) f_\pm \mp \gamma \left(f_-^2 - f_+^2\right) f_\pm = 0, \text{ for } 0 < r < R, \\ f_\pm(R) = \tfrac{1}{\sqrt{2}} \\ f_\pm(0) = 0 \text{ if } n_\pm \neq 0,\ f_\pm'(0) = 0 \text{ if } n_\pm = 0. \end{aligned}\right\} \quad (15)$$ ODE_R

The existence of such a solution follows easily, for example, by minimization of the energy

$$E^R_{n_+,n_-}(f_+, f_-) = \frac{1}{2}\int_0^R \left\{ \sum_{i=\pm} \left[(f_i')^2 + \frac{n_\pm^2}{r^2} f_i^2 \right] + \frac{1}{2}\left[(f_+^2 + f_-^2 - 1)^2 + \gamma (f_-^2 - f_+^2)^2 \right] \right\} r\, dr, \quad (16)$$ ER

over Sobolev functions satisfying the appropriate boundary conditions at $r = 0$ and $r = R$. Denote by $[f_{R,+}(r), f_{R,-}(r)]$ any solution of (15). As in the proof of (i) of Lemma 2.3 we have the simple *a priori* bound $|\Psi|^2 = (f_+^2(r) + f_-^2(r)) \leq 1$ for any solution to (16). By standard elliptic estimates, there exists a subsequence $R_n \to \infty$ for which the solutions $[f_{R,+}, f_{R,-}] \to [f_{\infty,+}, f_{\infty,-}]$ in $C^{1,\alpha}_{loc}[0,\infty)$, and the limit functions $[f_+^\infty(r), f_-^\infty(r)]$ give (weak) solutions to the ODE on $(0,\infty)$ with the same boundary condition at $r = 0$. By standard estimates we obtain the behavior $f_{\infty,\pm} \sim r^{n_\pm}$ near $r = 0$, and therefore $\psi_\pm(x) = f_{\infty,\pm}(r)e^{in_\pm\theta}$ is regular at $x = 0$ and solves (1–2) in $\mathbf{R}^2$.

On the other hand, by the Strong Maximum Principle, we have either $f_\pm > 0$ in $(0,\infty)$, or $f_\pm \equiv 0$ in $(0,\infty)$. Therefore, in order to conclude, it suffices to establish (3). For this purpose we derive a Pohozaev identity: we multiply the equation of $f_{R,\pm}(r)$ by $r^2 f_{R,\pm}'(r)$ and integrate with respect to $r \in (0,R)$. We obtain:

$$R^2\left((f_{R,+}'(R))^2 + (f_{R,-}(R))^2\right) + \int_0^R \left[(1 - f_{R,+}^2 - f_{R,-}^2)^2 + \gamma(f_{R,-}^2 - f_{R,+}^2)^2\right] r\, dr = \frac{1}{2}(n_+^2 + n_-^2).$$

By uniform convergence on $[0,R_0]$ for any $R_0 > 0$ we have

$$\int_0^{R_0} \left[(1 - f_{\infty,+}^2 - f_{\infty,-}^2)^2 + \gamma(f_{\infty,-}^2 - f_{\infty,+}^2)^2\right] r\, dr \leq \frac{1}{2}(n_+^2 + n_-^2),$$

and so letting $R_0 \to \infty$ we recover the condition (3). This completes the existence part of Lemma 3.1.

To prove uniqueness we use the basic approach of Brezis & Oswald [BO]. Let $[n_+, n_-] \in \mathbf{Z}^2$ be given, and suppose $[f_+, f_-]$ and $[g_+, g_-]$ are two solutions of (14). Denote by $\Delta_r f := \frac{1}{r}\left(r\, f'(r)\right)'$ the Laplacian for radial functions. Then we have:

$$-\frac{\Delta_r f_+}{f_+} + \frac{\Delta_r g_+}{g_+} = -\left[(1+\gamma)\left(f_+^2 - g_+^2\right) + (1-\gamma)\left(f_-^2 - g_-^2\right)\right]. \qquad (17) \quad \boxed{\texttt{pluseq}}$$

$$-\frac{\Delta_r f_-}{f_-} + \frac{\Delta_r g_-}{g_-} = -\left[(1-\gamma)\left(f_+^2 - g_+^2\right) + (1+\gamma)\left(f_-^2 - g_-^2\right)\right]. \qquad (18) \quad \boxed{\texttt{minuseq}}$$

We then multiply (17) by $(f_+^2 - g_+^2)$ and (18) by $(f_-^2 - g_-^2)$, and integrate over $0 < r < \infty$. Since $\psi_\pm(x) = f_\pm(r)e^{in_\pm\theta}$ defines a solution of the system (1–2) satisfying the condition (3), the estimates of Proposition 2.1 and Corollary 2.4 hold for $f_\pm, g_\pm$. In particular the integrals converge, and we may integrate by parts. As in [BO] we obtain:

$$\begin{aligned}
0 &\le \int_0^\infty \left\{ \left| f_+' - \frac{f_+}{g_+} g_+' \right|^2 + \left| g_+' - \frac{g_+}{f_+} f_+' \right|^2 + \left| f_-' - \frac{f_-}{g_-} g_-' \right|^2 + \left| g_-' - \frac{g_-}{f_-} f_-' \right|^2 \right\} r\,dr \\
&= -\int_0^\infty \left\{ (1+\gamma)\left(f_+^2 - g_+^2\right)^2 + 2(1-\gamma)\left(f_+^2 - g_+^2\right)\left(f_-^2 - g_-^2\right) + (1+\gamma)\left(f_-^2 - g_-^2\right)^2 \right\} r\,dr \\
&\le -2\min\{1,\gamma\} \int_0^\infty \left[\left(f_+^2 - g_+^2\right)^2 + \left(f_-^2 - g_-^2\right)^2 \right] r\,dr,
\end{aligned}$$

since the quadratic form $(1+\gamma)X^2 + 2(1-\gamma)XY + (1+\gamma)Y^2 \ge 2\min\{1,\gamma\}(X^2+Y^2)$ is positive definite. Hence $f_\pm(r) = g\pm(r)$ for all $r \in (0,\infty)$, and we have proven uniqueness. $\diamondsuit$

We next present the proof of Theorem 1.3 on the monotonicity of the radial profiles. First, we define the spaces

$$X_0 := H^1((0,\infty); r\,dr),$$

$$X_n := \left\{ u \in X_0 : \int_0^\infty \frac{u^2}{r^2}\, r\,dr < \infty \right\}, \qquad \|u\|_{X_n}^2 = \int_0^\infty \left[(u')^2 + u^2 + \frac{n^2}{r^2}u^2 \right] r\,dr.$$

Of course the spaces X_n, $n \neq 0$ are all equivalent, but we define them this way for notational convenience. It is not difficult to show (see [ABG1]) that for $|n| \ge 1$, X_n is continuously embedded in the space of continuous functions on $(0,\infty)$ which vanish at $r = 0$ and as $r \to \infty$, and that $C_0^\infty((0,\infty))$ is dense in X_1. It is possible to define a global variational framework for the equivariant problems in affine spaces based on X_{n_+}, X_{n_-} to prove existence of solutions. The energy is the same as in (16), except it must be "renormalized" to prevent divergence of the $\frac{n_\pm^2}{r^2}$ term at infinity. Here we are only interested in the (formal) second variation of this renormalized energy,

$$D^2E_{n_+,n_-}(f_+, f_-)[u_+, u_-] := \int_0^\infty \left\{ \sum_{i=\pm} \left[(u_i')^2 + \frac{n_\pm^2}{r^2}u_i^2 + (f_+^2 + f_-^2 - 1)u_i^2 \right] \right.$$

$$+2(f_+u_+ + f_-u_-)^2 + 2\gamma(f_-u_- - f_+u_+)^2 \\ +\gamma(f_-^2 - f_+^2)(u_-^2 - u_+^2)\Big\}r\,dr,$$

defined for $[u_+, u_-] \in X_{n_+} \times X_{n_-}$.

We have the following remarkable fact about admissible radial solutions:

nondeg **Lemma 3.2** *For any $n_\pm \in \mathbf{Z}$, if $[f_+, f_-]$ is the (unique) admissible radial solution of (14),*

$$D^2E_{n_+,n_-}(f_+, f_-)[u_+, u_-] > 0 \quad \text{for all } [u_+, u_-] \in X_{n_+} \times X_{n_-} \setminus \{[0,0]\}.$$

In other words, the radial solutions are non-degenerate local minimizers of the renormalized energy. An analogous statement for the Ginzburg–Landau equation with magnetic field was derived in [ABG1], and this observation then became the main step in the proof of uniqueness of equivariant solutions proved there. The basic idea is that were there two admissible solutions to the equivariant vortex equations, each being a local minimizer of the energy there would be a third, non-minimizing solution via the Mountain–Pass Theorem. The argument was achieved by restriction to a convex constraint set (to eliminate the possibility of non-admissible solutions, which might not be local minimizers.) The method works because the constraints play the role of a sub- and super-solution pair for the Ginzburg–Landau equations, and hence the mountain pass solutions obtained would lie in the interior of the constraint set. Unfortunately, in our vector-valued case the sub-solution structure is not apparent and the argument does not seem to carry over.

Proof of Lemma 3.2: We follow [ABG1], and note the following calculus identity:

$$f^2(r)\left[\left(\frac{u(r)}{f(r)}\right)'\right]^2 = (u')^2 - 2\frac{uu'f'}{f} + u^2\frac{(f')^2}{f^2} = (u')^2 - \left(\frac{u^2}{f}\right)' f'.$$

Let $u_\pm \in C_0^\infty((0,\infty))$ (if $n_\pm = 0$, take $u_\pm \in C_0^\infty([0,\infty))$ instead.) Then $[\frac{u_+^2}{f_+}, \frac{u_-^2}{f_-}]$ gives an admissible test function in the weak form of the system (14),

$$\begin{aligned} 0 &= DE_{n_+,n_-}(f_+, f_-)\left[\frac{u_+^2}{f_+}, \frac{u_-^2}{f_-}\right] \\ &= \int_0^\infty \Big\{ \sum_{i=\pm}\left((u_i')^2 + \frac{n_i^2}{r^2}u_i^2 - f_i^2\left[\left(\frac{u_i}{f_i}\right)'\right]^2\right) \\ &\qquad +(f_+^2 + f_-^2 - 1)(u_+^2 + u_-^2) + \gamma(f_-^2 - f_+^2)(u_-^2 - u_+^2)\Big\}r\,dr. \end{aligned}$$

Rearranging, we obtain the useful identity,

$$\int_0^\infty \Big\{ \sum_{i=\pm}\left((u_i')^2 + \frac{n_i^2}{r^2}u_i^2\right) + (f_+^2 + f_-^2 - 1)(u_+^2 + u_-^2) + \gamma(f_-^2 - f_+^2)(u_-^2 - u_+^2)\Big\}r\,dr$$

$$= \int_0^\infty \sum_{i=\pm} f_i^2 \left[\left(\frac{u_i}{f_i} \right)' \right]^2 \geq 0.$$

We then substitute into the expression for the second variation (19):

$$\begin{aligned} D^2 E_{n_+,n_-}(f_+, f_-)[u_+, u_-] &= \int_0^\infty \left\{ \sum_{i=\pm} f_i^2 \left[\left(\frac{u_i}{f_i} \right)' \right]^2 + 2(f_+u_+ + f_-u_-)^2 + 2\gamma(f_-u_- - f_+u_+)^2 \right\} \\ &\geq \int_0^\infty \left\{ 2(f_+u_+ + f_-u_-)^2 + 2\gamma(f_-u_- - f_+u_+)^2 \right\}, \end{aligned} \quad (19) \quad \boxed{\texttt{2varbel}}$$

valid for all $u_\pm \in C_0^\infty((0,\infty))$ (or, $u_\pm \in C_0^\infty([0,\infty))$ if the respective $n_\pm = 0$.) The case of general $u_\pm \in X_1$ (or X_0, in case one of $n_\pm = 0$,) then follows by density. Clearly, $D^2 E_{n_+,n_-}(f_+, f_-) \geq 0$ (as a quadratic form.) If it were zero for some $[u_+, u_-]$, then we would have $f_+u_+ = f_-u_- = -f_+u_+$ almost everywhere. Since $f_\pm(r) > 0$ for $r > 0$, we conclude that the second variation is strictly positive definite, as claimed.

$\diamondsuit$

Proof of Theorem 1.3: Let $u_\pm(r) := f_\pm'(r)$. Differentiating the equation (14), we obtain

$$\begin{aligned} 0 &= -u_\pm'' - \frac{1}{r}u_\pm' + \frac{n_\pm^2}{r^2}u_\pm + (f_+^2 + f_-^2 - 1)u_\pm + 2(f_+u_+ + f_-u_-)f_\pm \\ &\quad \mp 2\gamma(f_-u_- - f_+u_=)f_\pm - \frac{2n_\pm^2}{r^3}f_\pm + \frac{1}{r^2}u_\pm. \end{aligned} \quad (20) \quad \boxed{\texttt{ueq}}$$

We observe that all but the last two terms form part of the second variation of energy, (19). Define

$$v_\pm = \min\{0, u_\pm\} \leq 0, \qquad w_\pm = \max\{0, u_\pm\} \geq 0.$$

First, assume $\gamma \geq 1$. We multiply the respective equation in (20) by $v_\pm$, (and use $v_+w_+ = 0$ and $v_-w_- = 0$,) and integrate by parts. Note that by conclusion (ii) of Lemma 3.1, $f_\pm(r) > 0$ for all r and $f_\pm(r) \sim r^{n_\pm}$ for r near zero. Therefore, if $n_\pm \geq 1$, $u_\pm(r) = f_\pm'(r) > 0$ in some neighborhood $r \in (0,\delta)$. Thus, in case $n_\pm \geq 1$, $v_\pm$ is supported away from $r = 0$. By the asymptotic estimates of Corollary 2.4 we may then conclude that $v_\pm \in X_{n_\pm}$. Furthermore,

$$\int_0^\infty v_\pm \left(u_\pm'' + \frac{1}{r}u_\pm' \right) r\,dr = -\int_0^\infty (v_\pm')^2\, r\,dr,$$

with no boundary terms. In case $n_\pm = 0$, then $u_\pm \in X_0$ by the regularity of solutions, and $u_\pm(0) = f_\pm'(0) = 0$. The integration by parts formula above again holds with no boundary condition in this case as well. Combining terms and recognizing that many terms form part of the second variation of energy (19), we obtain:

$$\begin{aligned} 0 &= D^2 E_{n_+,n_-}(f_+, f_-)[v_+, v_-] + 2(1-\gamma) \int_0^\infty f_+\, f_-(w_-\, v_+ + v_-\, w_+)\, r\,dr \\ &\quad + \sum_{i=\pm} \int_0^\infty \left[\frac{1}{r^2}v_i^2 - \frac{2n_i^2}{r^3} f_i\, v_i \right] r\,dr. \end{aligned}$$

Each term above has a sign, and we obtain

$$0 \le D^2 E_{n_+,n_-}(f_+, f_-)[v_+, v_-] \le -\int_0^\infty \frac{1}{r^2}(v_+^2 + v_-^2)\, r\, dr < 0,$$

a contradiction to Lemma 3.2 unless both $v_\pm \equiv 0$, that is unless $f'_\pm(r) \ge 0$ for all $r > 0$. This proves (i).

Now assume $0 < \gamma < 1$, $n_- = 0$ and $n_+ \ge 1$. This time we multiply the equation for u_+ by v_+ and the equation for u_- by w_-, and again integrate by parts. Just as in the previous case, $w_- \in X_0$, and the boundary term in the integration will all vanish. This time we obtain:

$$\begin{aligned} 0 &= D^2 E_{n_+,n_-}(f_+, f_-)[v_+, w_-] + 2(1-\gamma)\int_0^\infty f_+ f_-(v_+ v_- + w_+ w_-)\, r\, dr \\ &\quad + \int_0^\infty \left(\frac{1}{r^2}(v_+^2 + w_-^2) - \frac{2n_+^2}{r^3} f_+ v_+ \right) r\, dr. \end{aligned}$$

Since $f_+ > 0$, $v_+ \le 0$ and $v_+ v_-, w_+ w_- \ge 0$, we conclude

$$D^2 E_{n_+,n_-}(f_+, f_-)[v_+, w_-] \le -\int_0^\infty \frac{1}{r^2}(v_+^2 + w_-^2)\, r\, dr < 0,$$

a contradiction with Lemma 3.2 unless $v_+ \equiv 0$ and $w_- \equiv 0$. That is, unless $f'_+(r) \ge 0$ and $f'_-(r) \le 0$ for all $r > 0$.
$\diamondsuit$

4 Another approach to uniqueness

We give a second proof of the uniqueness of the equivariant solutions

$$[\psi_+, \psi_-] = \left[f_+(r) e^{in_+\theta}, f_-(r) e^{in_-\theta} \right]$$

which is based on an extension of Krasnoselskii's method [K] to variational elliptic systems.

For a vector $u = (u_1, \dots, u_m) \in \mathbf{R}^m$, we say u is positive, and write $u > 0$, if $u_i > 0$ for all $i = 1, \dots, m$. We denote by $u^2 = (u_1^2, \dots, u_m^2)$.

Kras **Theorem 4.1** *Suppose $G : \Omega \times \mathbf{R}^m \to \mathbf{R}$, and $G(x,u)$ is strictly convex in $u > 0$ for every fixed $x \in \Omega \subset \mathbf{R}^n$. Then, there is at most one positive solution u to*

$$\begin{cases} -\Delta u_j + \partial_{u_j} G(x, u^2) u_j = 0, & x \in \Omega, \ j = 1, \dots, m \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{21}$$

Ke

Proof: Define the energy associated to this problem,

$$E(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + G(x,u^2)\right)$$

for $u \in H^1_0(\Omega;\mathbf{R}^m)$. Suppose u, v are both positive solutions of (21), and define w_j so that $v_j(x) = u_j(x)w_j(x)$ for all $j = 1,\dots,m$. By the Hopf boundary lemma, $w \in C^1(\overline{\Omega})$. Multiplying the equation for u_j by $\frac{1}{2}u_j(w_j^2-1)$ and integrating by parts, we arrive at the identity

$$\frac{1}{2}\int_\Omega \left(|\nabla u_j|^2(w_j^2-1) + 2u_jw_j\nabla u_j\cdot\nabla w_j\right) = -\frac{1}{2}\int_\Omega u_j^2(w_j^2-1)\partial_{u_j}G(x,u^2). \qquad (22)$$ id

Next, we expand the energy of v, using the above identity:

$$\begin{aligned}
E(v) &= E(u_1w_1,\dots,u_mw_m)\\
&= \frac{1}{2}\sum_{j=1}^m\int_\Omega \left(w_j^2|\nabla u_j|^2 + 2u_jw_j\nabla u_j\cdot\nabla w_j + u_j^2|\nabla w_j|^2 + G(x,v^2)\right)\\
&= \frac{1}{2}\sum_{j=1}^m\int_\Omega \left(|\nabla u_j|^2 - u_j^2(w_j^2-1)\partial_{u_j}G(x,u^2) + u_j^2|\nabla w_j|^2 + G(x,v^2)\right)\\
&= E(u) + \frac{1}{2}\int_\Omega u_j^2|\nabla w_j|^2 + \frac{1}{2}\int_\Omega\left[G(x,v^2) - G(x,u^2) - \sum_{j=1}^m(v_j^2-u_j^2)\partial_{u_j}G(x,u^2)\right].
\end{aligned}$$

By the strict convexity of G, we have

$$G(x,t) - G(x,s) - \sum_{j=1}^m (t_j - s_j)\partial_{s_j}G(x,s) \ge 0$$

for all $x \in \Omega$ and for all $s, t > 0$ in $\mathbf{R}^m$, with equality if and only if $s = t$. In particular, if $u \not\equiv v$, we have $E(v) > E(u)$. Reversing the roles of the variables u and v we also see $E(u) > E(v)$, a contradiction unless $u = v$.
$\diamondsuit$

general **Remark 4.2** By the same proof, we obtain uniqueness for the more general semilinear variational system,

$$\begin{cases} -\text{div}\ (A_j(x)\nabla u_j) + \partial_{u_j}G(x,u^2)u_j = 0, & x\in\Omega,\ j = 1,\dots,m\\ u > 0, & x \in \Omega,\\ u = 0, & x\in\partial\Omega,\end{cases}$$

with $A_j(x)$ symmetric, $n\times n$ elliptic matrix-valued functions in Ω.

The vortex profiles $f_\pm(r)$ being defined on the semi-infinite interval $r \in [0, \infty)$, we must modify this basic uniqueness theorem to fit this setting. In particular, the energy associated to the equivariant vortices is infinite on the entire interval. However, by the basic estimates proven in Proposition 2.1, the *difference* between the energies of two solutions will converge.

In this setting, we let $u = (f_+(r), f_-(r))$ and

$$G(r, s, t) = \frac{n_+^2}{r^2} s + \frac{n_-^2}{r^2} t + \frac{1}{2} (1 - s - t)^2 + \frac{\gamma}{2} (s - t)^2 .$$

It is easy to verify that this G is strictly convex in $(s, t) > 0$ for each $r \in [0, \infty)$; indeed, its Hessian is the constant matrix

$$D^2_{(s,t)} G(r, s, t) = \begin{bmatrix} 1+\gamma & 1-\gamma \\ 1-\gamma & 1+\gamma \end{bmatrix} .$$

The equations (15) for $f_\pm(r)$ take the form

$$\begin{cases} -\Delta_r f_j + \partial_{f_j} G(r, f_+^2, f_-^2) f_j = 0, & r \in [0, \infty), \ j = +, - \\ f_\pm(r) > 0, & r \in [0, \infty), \\ f_\pm(r) \to \frac{1}{\sqrt{2}}, & r \to \infty. \end{cases} \tag{23}$$

Ve

Recall from the proof of Lemma 3.1 that $f_\pm(r) \sim r^{|n_\pm|}$ as $r \to 0$ when $n \neq 0$, and if $n_\pm = 0$, we have $f_\pm'(0) = 0$. We recall also the localized energies in $r \in [0, \infty)$ defined by (16), which take the form

$$E^R_{n_+, n_-}(f_+, f_-) = \frac{1}{2} \int_0^R \left\{ \sum_{j=+,-} (f_j'(r))^2 + G(r, f_+^2, f_-^2) \right\} r \, dr$$

Now assume that there are two such solutions, $u = (f_+, f_-) > 0$ and $v = (g_+, g_-) > 0$, and as above we let w be chosen with $v_j = u_j w_j$, $j = +, -$. By Lemma 3.1 we have $w \in C^1[0, \infty)$ and uniformly bounded. Because we no longer have a Dirichlet condition at $r = R$, the identity (22) takes the form:

$$\begin{aligned} &\frac{1}{2} \int_0^R \left((f_j')^2 (w_j^2 - 1) + 2 f_j, w_j \, f_j' w_j' \right) r \, dr \\ &\quad = -\frac{1}{2} \int_0^R f_j^2 (w_j^2 - 1) \partial_{f_j} G(r, f_+^2, f_-^2) \, r \, dr + r f_j'(r) f_j(r) (w_j^2(r) - 1) \Big|_0^R \\ &\quad = -\frac{1}{2} \int_0^R f_j^2 (w_j^2 - 1) \partial_{f_j} G(r, f_+^2, f_-^2) \, r \, dr + o(1), \end{aligned}$$

for $j = +, -$ and as $R \to \infty$, where we have used Proposition 2.1 to estimate the boundary term at $r = R \to \infty$ and Lemma 3.1 to eliminate the term at $r = 0$. As in the proof of

Theorem 4.1 we compare the energies using the above identity,

$$\begin{aligned}
E^R_{n_+,n_-}&(g_+,g_-) - E^R_{n_+,n_-}(f_+,f_-) \\
&= E^R_{n_+,n_-}(v) - E^R_{n_+,n_-}(u) \\
&= \frac{1}{2}\int_0^R \sum_{j=+,-} u_j^2 (w_j')^2 r\,dr + o(1) \\
&+\frac{1}{2}\int_0^R \left[G(r,v^2) - G(r,u^2) - \sum_{j=+,-}(v_j^2-u_j^2)\partial_{u_j} G(r,u^2)\right] r\,dr.
\end{aligned}$$

By the estimates of Proposition 2.1, the term

$$\frac{n_\pm^2}{r^2}(f_\pm^2 - g_\pm^2)$$

is integrable, as are all the other terms which appear in the energy density. Hence the left-hand side converges as $R\to\infty$ and using Fatou's Lemma, we conclude

$$\begin{aligned}
&\lim_{R\to\infty}(E^R_{n_+,n_-}(g_+,g_-) - E^R_{n_+,n_-}(f_+,f_-)) \geq \\
&\quad \frac{1}{2}\int_0^\infty \left[G(r,v^2) - G(r,u^2) - \sum_{j=+,-}(v_j^2-u_j^2)\partial_{u_j} G(r,u^2)\right] r\,dr > 0,
\end{aligned}$$

unless $u=v$, by the strict convexity of $G(r,s,t)$ in $(s,t)>0$. Reversing the role of u and v, we arrive at a contradiction unless $u=v$.

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